Then $\cot\beta = c/\sqrt{(d^2-c^2)}$.

$\cot\beta_1 = e\cot\beta = ec/\sqrt{(d^2-c^2)}$.

$\cot\beta_2 = e\cot\beta_1 = e^2c/\sqrt{(d^2-c^2)}$.

....

$\cot\beta_n = e^n c\sqrt{(d^2-c^2)} = x/\sqrt{(d^2-x^2)}$.

$\therefore\ xcde^n/\sqrt{[d^2-c^2(1-e^{2n})]}$.

If $e=1$, $x=c$.

AVERAGE AND PROBABILITY.

102. Proposed by PROFESSOR CAVALLIN.

A random straight line is determined by two points taken at random within a sphere; find the average velocity acquired by a particle in descending the line. [No. 6742, *Educational Times*. Unsolved.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let (x, y, z), (u, v, w) be the coördinates of the two random points with center of sphere as origin. Let a=radius, $\sqrt{(a^2-x^2)}=y'$, $\sqrt{(a^2-u^2)}=v'$, $\sqrt{(a^2-x^2-y^2)}=z'$, $\sqrt{(a^2-u^2-v^2)}=w'$. The elevation of the one end of the line above the other$=(u-x)$.

Velocity$=\sqrt{[2g(u-x)]}$. The limits of x are $-a$ and a; of u, x and a and doubled for the case when $u<x$; of y, $-y'$ and y'; of z, $-z'$ and z'; of v, $-v'$ and v'; of w, $-w'$ and w'. Then since $(\frac{4}{3}\pi a^3)^2$ is the number of ways the two points can be taken, we get

$$\triangle = \frac{2}{(\frac{4}{3}\pi a^3)^2}\int_{-a}^{a}\int_{x}^{a}\int_{-y'}^{y'}\int_{-z'}^{z'}\int_{-v'}^{v'}\int_{-w'}^{w'}\sqrt{[2g(u-x)]}\,dxdudydzdvdw$$

$$=\frac{9\sqrt{2g}}{8a^6}\int_{-a}^{a}\int_{x}^{a}\sqrt{(u-x)}(a^2-x^2)(a^2-u^2)dxdu$$

$$=\frac{3\sqrt{2g}}{35a^6}\int_{-a}^{a}(5a+2x)(a^2-x^2)(a-x)^{\frac{5}{2}}\,dx$$

Let $x=a\cos\theta$.

$$\therefore\ \triangle = \frac{192\sqrt{ag}}{35}\int_{0}^{\pi}(7-4\sin^2\tfrac{1}{2}\theta)\sin^8\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta d\theta = \frac{256\sqrt{ag}}{273}.$$

103. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

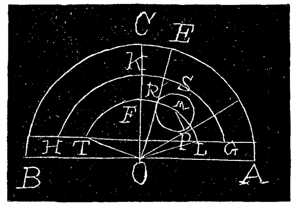
A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a

radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

I. **Solution by the PROPOSER.**

Let ACB be the given semi-circle, PRS the random circle, O and M their respective centers, OD and OE tangent radii to the circle PSR, OF and CK each equal MP, OC perpendicular to AB, GH parallel to AB, GKH an arc of a circle whose center is O, and LIT an arc of a circle through M and whose center is at O.



Put $MP=OF=CK=x$, $OM=y$, $OA=1$, $\angle ROP=\theta$, arc $LIT=\phi$. Then we have $OK=1-x$, $\theta=2\sin^{-1}(x/y)$, and $\phi=[\pi-2\sin^{-1}(x/y)]$.

Now since the center of the circle PRS may be anywhere in the segment GKH, the limits of y are x and $1-x$; and those of x are 0 and $\frac{1}{2}$. Hence, the required chance is

$$p=\frac{\int_0^{\frac{1}{2}}\int_x^{1-x}\theta\phi\,dx\,dy}{\int_0^{\frac{1}{2}}\int_x^{1-x}\pi\phi\,dx\,dy}$$

$$=\int_0^{\frac{1}{2}}\Bigg[\pi(1-x)^2\sin^{-1}\frac{x}{1-x}-2(1-x)^2\left[\sin^{-1}\frac{x}{1-x}\right]^2$$

$$-4x(1-2x)^{\frac{1}{2}}\sin^{-1}\frac{x}{1-x}+\pi x(1-2x)^{\frac{1}{2}}$$

$$-4x^2\log\left(\frac{1-x}{x}\right)\Bigg]dx\div\pi\int_0^{\frac{1}{2}}\Bigg[\tfrac{1}{2}\pi(1-n)^2$$

$$-(1-x)^2\sin^{-1}\frac{x}{1-x}-x(1-2x)^{\frac{1}{2}}\Bigg]dx=\Bigg[\tfrac{2}{3}(1-x)^3\left(\sin^{-1}\frac{x}{1-x}\right)^2$$

$$-(\pi/3)(1-x)^3\sin^{-1}\frac{x}{1-x}+\left[\tfrac{1}{3}(1-2x)^{\frac{1}{2}}+\tfrac{8}{9}(1-2x)^{\frac{3}{2}}-\tfrac{1}{3}(1-2x)^{\frac{5}{2}}\right]\sin^{-1}\frac{x}{1-x}$$

$$-\frac{\pi}{12}(1-2x)^{\frac{1}{2}}-\frac{2\pi}{9}(1-2x)^{\frac{3}{2}}+\frac{\pi}{12}(1-2x)^{\frac{5}{2}}$$

$$-\tfrac{4}{9}x+\tfrac{4}{9}\log(1-x)-\tfrac{4}{3}x^3\log\left(\frac{x}{1-x}\right)\Bigg]_0^{\frac{1}{2}}$$

$$\div \pi \Big[\tfrac{1}{3}(1-x)^3 \sin^{-1}\frac{x}{1-x} - \frac{\pi}{6}(1-x)^3$$

$$+\tfrac{1}{12}(1-2x)^{\frac{1}{2}} + \tfrac{2}{9}(1-2x)^{\frac{3}{2}} - \tfrac{1}{12}(1-2x)^{\frac{5}{2}} \Big]_0^{\frac{1}{2}}$$

$$=\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

MISCELLANEOUS.

FURTHER REMARK ON PROBLEM 90.

The results of the problem may be put in a better form as follows : e must be a function of s, say $e=f(s)$. For a *continuous* x,

$$f(s+x)=f(s)+x\Delta f(s)+\frac{x(x-1)}{2\,!}\Delta^2 f(s)+\frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(s).$$

Put $s=0$, and substitute for $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, their values 21, $\frac{7}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, respectively, and

$$f(x)=21+\tfrac{41}{12}x+\frac{x^3}{12};$$

replacing x by a *continuous* s,

$$e=f(s)=21+\frac{41s}{12}+\frac{s^3}{12},$$

which determines once for all the functional relation between any value of s and e. This result is somewhat analagous to the primitive functional relation found from a differential equation, only here difference coefficients enter instead of differential coefficients, and the work is infinitely simpler. Of course the same results could have been obtained by La Grange's formula of interpolation. In order to use the above formula for calculating e for any distance s, 100 yards must be taken as the unit.

E. D. ROE, JR.

93. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Prove that $-(\sqrt{-1})^{\sqrt{-1}}=e^{(\sqrt{-1}-\frac{1}{2})\pi}$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

In the Napierian system of logarithms we always have $a^n=e^{n\log a}$....(1).